CMS 2: 213–228 (2005)

DOI: 10.1007/s10287-005-0024-2

Design of induction motors using a mixed-variable approach*

Giampaolo Liuzzi[1], Stefano Lucidi[1], Veronica Piccialli[1], Marco Villani[2]

[1] Università di Roma "La Sapienza", Dipartimento di Informatica e Sistemistica, Via Buonarroti 12, 00185 Roma, Italy (e-mail: {liuzzi,lucidi,piccialli}@dis.uniroma1.it
[2] Università de L'Aquila, Dipartimento di Ingegneria Elettrica, Poggio di Roio, 67040 L'Aquila, Italy (e-mail: villani@ing.univaq.it)

Abstract. In this paper we are concerned with the problem of optimally designing three-phase induction motors. This problem can be formulated as a mixed variable programming problem. Two different solution strategies have been used to solve this problem. The first one consists in solving the continuous nonlinear optimization problem obtained by suitably relaxing the discrete variables. On the opposite, the second strategy tries to manage directly the discrete variables by alternating a continuous search phase and a discrete search phase. The comparison between the numerical results obtained with the above two strategies clearly shows the fruitfulness of taking directly into account the presence of both continuous and discrete variables.

Keywords: Optimal design, Induction Motors, Mixed-variable optimization, Derivative-free methods

1 Introduction

Three-phase induction motors are widely used in industrial applications, and have a significant impact on electricity consumption.

The European Committee of Manufacturers of Electrical Machines and Power Electronics and the European Commission stated a joint classification scheme that will enable customers and users of induction motors to have a simple efficiency ranking of these components. The classification scheme has only three efficiency

* This work was supported by CNR/MIUR Research Program "*Metodi e sistemi di supporto alle decisioni*", Rome, Italy.

classes, namely a "standard efficient" (eff1), an "energy efficient" (eff2) and a "high (premium) efficient" (eff3) class.

This classification needs to develop new ranges of motors. The design of "high efficiency" induction motors requires the use of specific optimization techniques. The growing demand of high-performance motors requires the definition of more and more efficient designs. The only way to obtain such kinds of motors is to use automatic optimization procedures along with the definition of an analytical model of the motor itself. Such a model can be obtained by reducing the physical description of the motor to an equivalent representation as resistances and inductances [2]. The adopted analytical model takes into account the influence of saturation on stator and rotor reactances, the influence of skin effect on rotor parameters and the effects of the temperature raising on motor resistances.

In Sect. 2 we describe the optimal design problems which arise in the design of induction motors. Such problems can be naturally stated as mixed-variable programming problems. Section 3 is devoted to a continuous approach used for solving the problem. In Sect. 4 we introduce the mixed-variable programming algorithm which has been used to tackle the optimal design problem. Finally, in Sect. 5 we describe the results obtained by using the mixed-variable approach.

In the paper we denote by $|\cdot|$ the cardinality of a set.

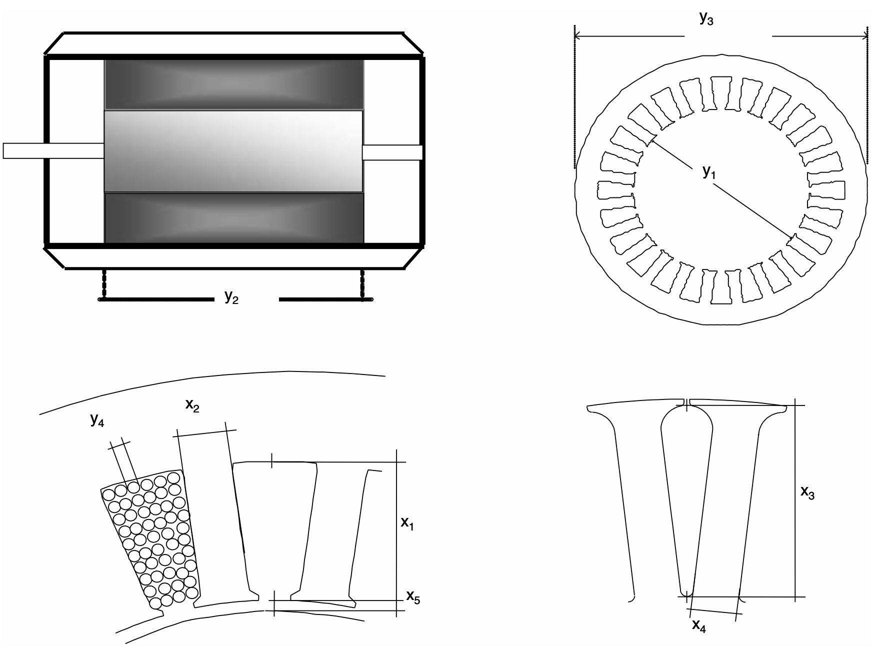
2 Problem description

The optimal design of electric motors requires particular attention in the choice of the objective function that usually concerns economic or performance features. In order to cover both these aspects of the design problem, we have chosen two objective functions that can affect the design optimization of three-phase induction motors. Particularly:

$f^{(1)}$: Manufacturing cost in Euros (to be minimized);
$-f^{(2)}$: Rated efficiency (to be maximized).

The induction motor is completely determined by the following independent variables which define the stator and rotor dimensions. They are:

the stator slot height in mm	(x_1)
the stator tooth width in mm	(x_2)
the rotor slot height in mm	(x_3)
the rotor tooth width in mm	(x_4)
the air-gap length in mm	(x_5)
the air-gap flux density in Tesla	(x_6)
the stack length in mm	(y_2)
the outer stator diameter in mm	(y_3)
the stator wire size in mm^2	(y_4)
the electrical steel type	(t)



Our aim is to design a motor without affecting heavily the tooling costs and the building process. For this reason some of the above quantities should assume a finite number of values. This is essentially due to the fact that the preexistent lamination punch tools or stator housing tools allow to handle only some prefixed values of the independent variables. Obviously, if we want to change all motor dimensions and renew the lamination tooling, then these kind of limitations can be neglected. In the former case however, the corresponding variables must be considered as discrete variables whose feasible values are related to the base components availability and to the limitations of the existing manufacturing process. As for the continuous variables, their values must be within given bounds which are connected to mechanical and technological constraints, according to the manufacturer suggestions.

The variable t represents the electric steel type that plays a significant role on the motor performance: its right choice, combined with the design optimization of the motor, should allow to achieve better results and higher efficiency. The choice of a "suitable" electric steel type depends on several aspects such as cost, workability, "business tradition" and storehouse demands. In this study six "fully processed" commercial steels have been considered, labelled with 0, 1, ..5 (where $t = 0$ represents an "high performance" and "high cost" steel type while $t = 5$ represents a "low performance", "low cost" steel type).

Beside the bound constraints on the variables, the problem involves also some nonlinear constraints which concern mainly the motor performances. In particular, they are: the stator winding temperature, the rotor bars temperature, the flux density in the stator and rotor teeth, the rated slip, the starting torque, the starting current, the breakdown torque, the power factor at rated load and the stator slot fullness.

Finally, depending on the choice of the objective function ($f^{(l)}$, $l = 1, 2$), we come up with the following mixed variable programming problems for a 7.5 kW, 4 pole, 380 V, 50 Hz, three-phase induction motor,

$$\begin{aligned} \min_{x,y,t}\ & f^{(l)}(x, y, t) \\ & g(x, y, t) \le 0 \\ & 16.0 \le x_1 \le 19.0 \\ & 4.5 \le x_2 \le 6.5 \\ & 16.0 \le x_3 \le 18.5 \\ & 3.5 \le x_4 \le 5.0 \\ & 0.3 \le x_5 \le 0.5 \\ & 0.5 \le x_6 \le 0.68 \\ & y_1 \in \{126.6, 131.6\} \\ & y_2 \in \{140, 150, 160, 170, 180, 190\} \\ & y_3 \in \{180, 200, 220\} \\ & y_4 \in \{1.4, 1.45, 1.5, 1.55, 1.5727, 1.6, 1.65, 1.7, 1.75\} \\ & t \in \{0, 1, 2, 3, 4, 5\}, \end{aligned} \tag{P^l}$$

where $x = (x_1\ x_2\ x_3\ x_4\ x_5\ x_6)^T$ and $y = (y_1\ y_2\ y_3\ y_4)^T$.

The distinguishing features of these problems are reported below.

(i) To evaluate the objective and constraint functions on a given point, it is necessary to perform a numerical simulation of the motor operating status. For this reason, an explicit representation of the objective and constraint functions is not available.

(ii) The constraints $g(x, y, t) \le 0$ are not very restrictive, namely, it is relatively easy to find a feasible point and to remain in the feasible region.

(iii) y and t can assume only a finite number of values. In particular, the discrete variable t affects the structure of the objective and constraint functions. Moreover, it cannot assume any intermediate value since for such values the corresponding optimization problem is undefined.

Taking into account property (ii), we can transform Problem (P^l) using a standard technique to eliminate the nonlinear constraints (see, for instance, [5]). In particular, we use these new objective functions.

$$\tilde{f}^{(l)}(x, y, t) = \begin{cases} f^{(l)}(x, y, t) & \text{if } g(x, y, t) \le 0 \\ +\infty & \text{otherwise} \end{cases} \qquad l = 1, 2.$$

Hence we consider problems

$$\begin{aligned} \min_{x,y,t} \ & \tilde{f}^{(l)}(x, y, t) \\ & l_x \le x \le u_x \\ & y_i \in Y_i, \quad i = 1, 2, 3, 4 \\ & t \in T, \end{aligned} \tag{$\tilde{P}^l$}$$

for $l = 1, 2$, where

$$l_x = \begin{pmatrix} 16.0 \\ 4.5 \\ 16.0 \\ 3.5 \\ 0.3 \\ 0.5 \end{pmatrix} u_x = \begin{pmatrix} 19.0 \\ 6.5 \\ 18.5 \\ 5.0 \\ 0.5 \\ 0.68 \end{pmatrix}$$

and

$$\begin{aligned} Y_1 &= \{126.6, 131.6\} \\ Y_2 &= \{140, 150, 160, 170, 180, 190\} \\ Y_3 &= \{180, 200, 220\} \\ Y_4 &= \{1.4, 1.45, 1.5, 1.55, 1.5727, 1.6, 1.65, 1.7, 1.75\} \\ T &= \{0, 1, 2, 3, 4, 5\}. \end{aligned}$$

3 Continuous approach

A first attempt consists in solving the following nonlinear continuous optimization problems

$$\begin{aligned} \min_{x,y} \ & \tilde{f}^{(l)}(x, y, t)|_{t=h} \\ & l_x \le x \le u_x \\ & l_y \le y \le u_y, \end{aligned} \tag{$\tilde{R}^l_h$}$$

for $l = 1, 2$, obtained by setting $t = h$, for all $h = 0, 1, \ldots, 5$ and relaxing, in a suitable way, the discrete variables y and where

$$l_y = \begin{pmatrix} 126.6 \\ 140.0 \\ 180.0 \\ 1.4 \end{pmatrix} u_y = \begin{pmatrix} 131.6 \\ 190.0 \\ 220.0 \\ 1.75 \end{pmatrix}.$$

Problems $(\tilde{R}_h^l)$ for $h = 0, 1, \ldots, 5$ and $l = 1, 2$ can be rewritten as

$$\begin{aligned} &\min_z \varphi(z) \\ &\quad l_z \le z \le u_z, \end{aligned} \tag{1}$$

where $z = (x, y) \in \Re^{n_z}$, $l_z = (l_x, l_y)$ and $u_z = (u_x, u_y)$.

By property (i) of Problem (P^l), an explicit representation of $\varphi(z)$ is not available. Hence, to solve Problem (1) we applied the derivative free algorithm proposed in [4] whose description is reported below.

Procedure DFA$(z^\circ, \alpha_{\text{tol}})$

Data. $\alpha^\circ > 0$.

1. Set $j = 1$.
2. Apply procedure $DF(n_z, z^j, \alpha^{j-1}, z^{j+1}, \alpha^j)$.
3. If $\alpha^j > \alpha_{\text{tol}}$ then set $j := j + 1$ and go to Step 2.
   else return (z^{j+1}, α^j).

We refer to [4] for the theoretical analysis of Procedure DFA.

Procedure DF$(n_z, z, \mu^0, \tilde{z}, \mu)$

Data. $\gamma > 0$, $\delta \in (0, 1)$, $\delta_1 \in (0, 1)$, $\theta \in (0, 1)$, $d^i = e^i$ and $\tilde{\alpha}^i = \mu^0$ for $i = 1, \ldots, n$.

1. *Initialization:* Set $i = 1$ and $z^i = z$.
2. *Direction choice:*
   - 2.1 Compute α^i_{max} s.t. $z^i + \alpha^i_{max} d^i = u^i_z$ and set $\alpha = \min\{\tilde{\alpha}^i, \alpha^i_{max}\}$.
     If $\alpha > 0$, $\varphi(z^i + \alpha d^i) \le \varphi(z^i) - \gamma(\alpha)^2$,
     then go to Step 4.
   - 2.2 Compute α^i_{max} s.t. $z^i - \alpha^i_{max} d^i = l^i_z$ and set $\alpha = \min\{\tilde{\alpha}^i, \alpha^i_{max}\}$.
     If $\alpha > 0$, $\varphi(z^i - \alpha d^i) \le \varphi(z^i) - \gamma(\alpha)^2$,
     then set $d^i = -d^i$ and go to Step 4.
3. *Direction failure:* Set $\bar{\alpha} = 0$, $\tilde{\alpha}^i = \theta\alpha$, and go to Step 5.
4. *Linesearch:*
   - 4.1 Let $\hat{\alpha} = \min\{\alpha^i_{max}, \frac{\alpha}{\delta}\}$.
     If $\alpha = \alpha^i_{max}$ or $\varphi(z^i + \hat{\alpha} d^i) > \varphi(z^i) - \gamma\hat{\alpha}^2$,
     then set $\bar{\alpha} = \alpha$, $\tilde{\alpha}^i = \alpha$ and go to Step 5.
   - 4.2 Set $\alpha = \hat{\alpha}$ and go to Step 4.1.
5. *New point:* Set $z^{i+1} = z^i + \bar{\alpha} d^i$.
6. *Stopping criterion:* If $i = n_z$, then $\tilde{z} = z^{i+1}$, $\mu = \max_{i=1,\ldots,n}\{\tilde{\alpha}^i, \delta_1\mu^0\}$
   else set $i = i + 1$ and go to Step 2.

Every problem has been solved starting from the initial point $z^\circ = (x^\circ, y^\circ)$ where

$$x^\circ = \begin{pmatrix} 17.5 \\ 5.5 \\ 17.8 \\ 4.0 \\ 0.4 \\ 0.5682 \end{pmatrix} \quad y^\circ = \begin{pmatrix} 126.6 \\ 160.0 \\ 200.0 \\ 1.5727 \end{pmatrix}.$$

that, for $t^\circ = 1$ represents a reference motor and whose objective function values are listed in the table below.

t°	$\tilde{f}^{(1)}(x^\circ, y^\circ, t^\circ)$	$\tilde{f}^{(2)}(x^\circ, y^\circ, t^\circ)$
0	179.174	−86.08%
1	174.396	−88.30%
2	169.547	−87.68%
3	164.691	−86.85%
4	159.840	−87.55%
5	154.943	−84.87%

Initial function values.

In the tables below we report the results obtained by using this strategy. In particular, in Table 1, for every value of the discrete variable t, we report the solution points of problems $(\tilde{R}_h^1)$ for $h = 0, 1, \ldots, 5$, along with their associated objective function values (manufacturing cost). It emerges that the best solution is obtained when $t = 5$ (low-cost electrical steel). In Table 2, again, we report the solution points of problems $(\tilde{R}_h^2)$ for $h = 0, 1, \ldots, 5$, along with their objective function values (rated efficiency) for every value of the discrete variable t. In this case, it emerges that the best solution is obtained when $t = 1$ (high-performance electrical steel).

Table 1. Solutions for Problem $(\tilde{R}_h^1)$, $h = 0, \ldots, 5$

t	0	1	2	3	4	5
x_1	16	16.005	16	16.016	16.021	16.016
x_2	6.5	6.5	6.5	5.879	6.5	6.5
x_3	16.519	16	16	16.8	16	16.175
x_4	4.016	4.016	4	4	4.508	4.062
x_5	0.4	0.4	0.4	0.4	0.4	0.4
x_6	0.584	0.568	0.568	0.568	0.568	0.568
y_1	126.6	126.631	126.6	126.698	126.675	126.784
y_2	140	140	142.891	140	154.58	151
y_3	194.951	189.656	191.375	189.5	189.5	189.5
y_4	1.5102	1.4	1.4	1.5727	1.4	1.5723
cost	165.054	156.603	154.551	152.82	147.945	146.571

Table 2. Solutions for Problem $(\tilde{R}_h^2)$, $h = 0, \ldots, 5$

t	0	1	2	3	4	5
x_1	16	16	16	16	16	16
x_2	6.5	6.5	6.5	5.812	6.5	6.5
x_3	18.5	18.5	18.5	18.5	18.5	18.5
x_4	3.922	3.767	3.99	3.914	3.702	3.802
x_5	0.319	0.3	0.354	0.337	0.3	0.356
x_6	0.568	0.568	0.567	0.568	0.568	0.566
y_1	126.6	131.596	127.476	128.35	129.334	126.6
y_2	158.859	158.25	163.109	161.773	155.873	169.594
y_3	220	220	220	220	220	220
y_4	1.75	1.75	1.75	1.75	1.75	1.75
efficiency	89.14%	91.14%	90.15%	89.39%	90.56%	88.07%

One possible way to obtain feasible points for the original problems (P^l), $l = 1, 2$, starting from the optimal solutions of problems $(\tilde{R}_h^l)$, $l = 1, 2$, $h = 0, 1, \ldots, 5$, consists in searching for the best rounded neighbors of the continuous solutions. In the following tables we report the best rounded neighbors of the solutions of problems $(\tilde{R}_h^l)$, $h = 0, \ldots, 5$ and $l = 1, 2$.

Table 3. Best rounded neighbors for Problem (P^1)

t	0	1	2	3	4	5
x_1	16	16.005	16	16.016	16.021	16.016
x_2	6.5	6.5	6.5	5.879	6.5	6.5
x_3	16.519	16	16	16.8	16	16.175
x_4	4.016	4.016	4	4	4.508	4.062
x_5	0.4	0.4	0.4	0.4	0.4	0.4
x_6	0.584	0.568	0.568	0.568	0.568	0.568
y_1	126.6	131.6	126.6	131.6	126.6	126.6
y_2	140	140	140	140	160	160
y_3	200	200	200	200	200	200
y_4	1.5	1.4	1.4	1.5727	1.4	1.55
cost	168.047	162.765	158.358	158.199	154.783	153.062

We note that this rounding strategy carries to a worsening of the objective function value and in one case even to the impossibility to get a feasible point (as pointed out by the '-' sign in Table 4), namely a point satisfying the nonlinear constraints. In particular, as concerns the manufacturing cost, for $t = 5$, it increases from 146.571 to 153.062 Euros.

As regards the rated efficiency, the situation is more stable in the sense that the efficiency of the rounded points is not so distant from the continuous ones but for $t = 5$ this strategy does not produce any feasible point. This stability is probably

Table 4. Best rounded neighbors for Problem (P^2)

t	0	1	2	3	4	5
x_1	16	16	16	16	16	–
x_2	6.5	6.5	6.5	5.812	6.5	–
x_3	18.5	18.5	18.5	18.5	18.5	–
x_4	3.922	3.767	3.99	3.914	3.702	–
x_5	0.319	0.3	0.354	0.337	0.3	–
x_6	0.568	0.568	0.567	0.568	0.568	–
y_1	126.6	126.6	126.6	131.6	126.6	–
y_2	160	160	160	170	160	–
y_3	220	220	220	220	220	–
y_4	1.75	1.75	1.75	1.75	1.75	–
efficiency	89.12%	90.96%	90.02%	89.33%	90.46%	–

due to the fact that the continuous solutions have two out of four variables (y_3 and y_4) which assume a discrete value.

4 A mixed-variable programming algorithm

Problem $(\tilde{P}^l)$ is a mixed variable programming problem. The presence of both continuous and discrete variables requires a suitable definition of a local minimum point, which is not immediate. In fact this notion refers to the behavior of the objective function in a "suitable neighborhood" of a given point. While a neighborhood of a continuous variable is well represented by a continuous ball, the neighborhood of a discrete variable must be defined taking into account the structure of the particular problem.

Following Audet and Dennis ([1]), we can characterize a local solution (x^*, y^*, t^*) of Problem $(\tilde{P}^l)$ as a point satisfying the following definition:

Definition 1 *A feasible point* (x^*, y^*, t^*) *is said to be a local minimizer of Problem* $(\tilde{P}^l)$ *with respect to the feasible discrete neighborhood* $\mathcal{N}(x^*, y^*, t^*)$ *if there exists an* $\epsilon > 0$ *such that* $\forall(\hat{x}, \hat{y}, \hat{t}) \in \mathcal{N}(x^*, y^*, t^*)$

$$\tilde{f}^{(l)}(x^*, y^*, t^*) \leq \tilde{f}^{(l)}(x, \hat{y}, \hat{t}) \qquad \forall x \in \mathcal{B}(\hat{x}, \epsilon) \cap [l_x, u_x], \tag{2}$$

where $\mathcal{N}(x^*, y^*, t^*)$ *is a finite set of feasible points.*

This definition implies that there are no better feasible solutions in the balls centered at the points belonging to the discrete neighborhood of (x^*, y^*, t^*). Note that this definition depends on the choice of the discrete neighborhoods, which hence represent a measure of the quality of the solution. In fact, a larger discrete neighborhood $\mathcal{N}(x^*, y^*, t^*)$ should give a better local minimizer, but this may increase the computational effort needed to locate the solution, so there is a trade off.

In [3] it has been introduced an algorithm for solving mixed variable programming problems based on the combination of a local search with respect to the continuous variables and of a local search in the discrete neighborhood of the current point. This algorithm has been applied to the solution of Problem $(\tilde{P}^l)$. In particular, it is based on the idea to alternate two phases:

- an attempt to update the continuous variables by a local continuous search (Phase 1) in $[l_x, u_x]$,
- an attempt to update the discrete variables by a local search in the discrete neighborhood of the current point (Phase 2).

Phase 1:
Given the current feasible point (x_k, y_k, t_k), the discrete variables are fixed to the value (y_k, t_k) and the following continuous optimization problem is considered:

$$\begin{aligned} &\min_x \tilde{f}^{(l)}(x, y_k, t_k) \\ &l_x \le x \le u_x. \end{aligned} \tag{3}$$

Starting from x_k, we perform an iteration of a derivative free local continuous search with the goal of finding a new vector $\tilde{x}_k$ which is, roughly speaking, a better approximation of a stationary point of Problem (3).

Phase 2:
In this phase we try to update the discrete variables by considering the points belonging to the discrete neighborhood $\mathcal{N}(\tilde{x}_k, y_k, t_k)$ of the point $(\tilde{x}_k, y_k, t_k)$ produced by Phase 1.

First, we simply evaluate the objective function at the points belonging to $\mathcal{N}(\tilde{x}_k, y_k, t_k)$. If one of these points produces a sufficient decrease from $\tilde{f}^{(l)}(\tilde{x}_k, y_k, t_k)$, then it becomes the current point, and a new iteration is performed.

If none of the points belonging to $\mathcal{N}(\tilde{x}_k, y_k, t_k)$ produces a sufficient decrease with respect to $\tilde{f}^{(l)}(\tilde{x}_k, y_k, t_k)$, we perform a further investigation, by selecting some of these points which can be considered promising. In particular, we still try to update the discrete variables by selecting some points belonging to $\mathcal{N}(\tilde{x}_k, y_k, t_k)$ with objective value not significantly worse than $\tilde{f}^{(l)}(\tilde{x}_k, y_k, t_k)$. Starting from each one of these points, we perform a suitable number of local continuous searches with the aim to obtain a point which produces a sufficient decrease from $\tilde{f}^{(l)}(\tilde{x}_k, y_k, t_k)$.

The proposed algorithm model is formally stated as follows:

Mixed Integer Variable Algorithm Model (MIVAM)

Data: $y_i^\circ \in Y_i, i = 1, \dots, 4, t^\circ \in T, x^\circ \in [l_x, u_x], \xi \geq 0, \theta \in (0, 1), \eta^0 > 0, \mu^{in} > 0.$

Step 0: Set $k = 0$, $\mu_k^0 = \mu^{in}$.

Step 1: Compute $\tilde{x}_k$ and μ_k by applying Procedure DF($n_x, x_k, \mu^0, \tilde{x}_k, \mu_k$) to Problem (1) where
$\varphi(z) = \tilde{f}^{(l)}(x, y, t)|_{y=y_k,\ t=t_k}$ and $z = x$. Set $\mu_{k+1}^0 = \mu_k$.

Step 2: If there exists a $(\hat{x}_{k+1}, \hat{y}_{k+1}, \hat{t}_{k+1}) \in \mathcal{N}(\tilde{x}_k, y_k, t_k)$ such that

$$\tilde{f}^{(l)}(\hat{x}_{k+1}, \hat{y}_{k+1}, \hat{t}_{k+1}) \leq \tilde{f}^{(l)}(\tilde{x}_k, y_k, t_k) - \eta_k,$$

set $x_{k+1} = \hat{x}_{k+1}$, $y_{k+1} = \hat{y}_{k+1}$,$t_{k+1} = \hat{t}_{k+1}$, $\eta_{k+1} = \eta_k$, and go to Step 5.

Step 3: Define $W_k = \{(x, y, t) \in \mathcal{N}(\tilde{x}_k, y_k, t_k) : \tilde{f}^{(l)}(x, y, t) \leq \tilde{f}^{(l)}(\tilde{x}_k, y_k, t_k) + \xi\}$.

3.1: If $W_k \neq \emptyset$, choose $(x', y'.t') \in W_k$, set $j = 1, x^j = x', \mu^{j-1} = \mu_k$.

Otherwise go to Step 4.

3.2: Compute x^{j+1} and μ^j by applying Procedure DF($n_x, x^j, \mu^{j-1}, x^{j+1}, \mu^j$) to Problem (1) where $\varphi(z) = \tilde{f}^{(l)}(x, y, t)|_{y=y',\ t=t'}$ and $z = x$.

3.3: If $\tilde{f}^{(l)}(x^{j+1}, y', t') \leq \tilde{f}^{(l)}(\tilde{x}_k, y_k, t_k) - \eta_k$, set $x_{k+1} = x^{j+1}$, $y_{k+1} = y'$, $t_{k+1} = t'$, $\eta_{k+1} = \eta_k$, and go to Step 5.

3.4: If $\mu^j > \mu_k$, set $j = j + 1$, and go to 3.2.

Otherwise set $W_k = W_k \setminus \{(x', y', t')\}$, and go to 3.1.

Step 4: Set $x_{k+1} = \tilde{x}_k$, $y_{k+1} = y_k$, $t_{k+1} = t_k$.
If $\tilde{f}^{(l)}(x_{k+1}, y_{k+1}, t_{k+1}) \leq \tilde{f}^{(l)}(x_k, y_k, t_k) - \eta_k$, set $\eta_{k+1} = \eta_k$.

Otherwise set $\eta_{k+1} = \theta\eta_k$.

Step 5: Set $k = k + 1$, and go to Step 1.

At Step 1 Phase 1 is performed by applying the local continuous search DF($n_x, x_k, \mu_k^0, \tilde{x}_k, \mu_k$). This procedure tries to produce a new point $\tilde{x}_k$, where the objective function is sufficiently decreased. In particular, if the procedure DF is not able to produce a sufficient decrease of the objective function, the point $\tilde{x}_k$ is set equal to x_k.

Phase 2 is performed in Steps 2 and 3. In particular, at Step 2 the objective function is evaluated at each point in $\mathcal{N}(\tilde{x}_k, y_k, t_k)$. If one of these points produces a decrease with respect to $f(\tilde{x}_k, y_k, t_k)$ greater or equal to η_k, then it becomes the current point and a new iteration is performed. Otherwise the discrete neighborhood is further investigated in Step 3. In particular, a set $W_k \subseteq \mathcal{N}(\tilde{x}_k, y_k, t_k)$ of points with objective value not significantly worse than $f(\tilde{x}_k, y_k, t_k)$ is selected. Each of these points $(x', y', t') \in W_k$ is considered promising, and the algorithm tries to determine if it is worth replacing y_k with y'. In particular, starting from each point $(x', y', t') \in W_k$, the local continuous search is repeated until

(a) it finds a point significantly better than $(\tilde{x}_k, y_k, t_k)$, or
(b) the test at Step 3.4 fails.

In case (a), the new point becomes the current iterate (with new discrete variables y', t'), and a new iteration is performed. In case (b), we reject the discrete variables y', t' because a sufficient decrease of the objective function has not been achieved.

At Step 4 the point $(\tilde{x}_k, y_k, t_k)$ becomes the new current point and, if neither the local continuous search nor the discrete search have been able to produce a decrease of the objective function greater or equal to η_k, then this parameter is reduced before starting the next iteration.

For an analysis of the theoretical properties of Algorithm MIVAM we refer to [3] where global convergence of algorithm MIVAM toward a stationary point of the mixed variable problem is proved provided that the objective function is smooth with respect to the continuous variables.

5 Numerical results

In this section we report the numerical results obtained by applying algorithm MIVAM (described in the previous section) to problems (P^l), $l = 1, 2$. To use this strategy it is necessary to define the discrete neighborhood of a given feasible point and the searching strategy within this neighborhood. In particular, we have used five different neighborhood definitions and two different search rules.
Given a feasible point, we only allow each component of the discrete variables to change from their actual values either to the previous or to the next ones. If a discrete variable is on its upper bound, the next value is assumed to be its lower bound. Conversely, if a discrete variable is on its lower bound, the previous value is assumed to be its upper bound. The continuous variables are left unchanged, unless the resulting point is unfeasible, in which case they are randomly generated in such a way to guarantee feasibility.

More precisely, let

$$Y_i = \{y_i^1, \ldots, y_i^{|Y_i|}\}, \quad i = 1, 2, 3, 4$$
$$T = \{t^1, \ldots, t^{|T|}\},$$

be the sets defined in Sect. 2. We introduce the following bijective functions $\sigma_i : Y_i \mapsto \{1, 2, \ldots, |Y_i|\}$, $i = 1, \ldots, 4$ and $\sigma_t : T \mapsto \{1, 2, \ldots, |T|\}$ such that

$$\sigma_i(y_i^h) = h \tag{4}$$
$$\sigma_t(t^h) = h. \tag{5}$$

Now, given a feasible point $w = (x, y, t)$, we consider the set $\mathcal{S}(w) = \{\hat{w} = (\hat{x}, \hat{y}, \hat{t})\}$ where $\hat{x}, \hat{y}, \hat{t}$ are such that

$$\begin{cases} |\sigma_i(\hat{y}_i) - \sigma_i(y_i)| \leq 1 & \text{if } 1 < \sigma_i(y_i) < |Y_i| \\ \sigma_i(\hat{y}_i) = |Y_i| & \text{if } \sigma_i(y_i) = 1 \\ \sigma_i(\hat{y}_i) = 1 & \text{if } \sigma_i(y_i) = |Y_i| \end{cases} \tag{6}$$

$$\begin{cases} |\sigma_t(\hat{t}) - \sigma_t(t)| \leq 1 & \text{if } 1 < \sigma_t(t) < |T| \\ \sigma_t(\hat{t}) = |T| & \text{if } \sigma_t(t) = 1 \\ \sigma_t(\hat{t}) = 1 & \text{if } \sigma_t(t) = |T| \end{cases} \tag{7}$$

We define the distance $d(w, \hat{w})$ of a point $\hat{w}$ from w as the number of discrete components of $\hat{w}$ which are different from those of w, namely

$$d(w, \hat{w}) = \left| \bigcup_{i=1}^{4} \{\hat{y}_i : \hat{y}_i \neq y_i\} \cup \{\hat{t} : \hat{t} \neq t\} \right|.$$

Then, we define five different discrete neighborhoods obtained by considering points $\hat{w} \in \mathcal{S}(w)$ such that $d(w, \hat{w}) \leq \bar{d}$, namely

$$\mathcal{N}_{\bar{d}}(w) = \{\hat{w} \in \mathcal{S}(w) : d(w, \hat{w}) \leq \bar{d}\}$$

where $\bar{d} = 1, \ldots, 5$.

As regards the search rules within the discrete neighborhood at Steps 2 and 3.1, we take into consideration two possibilities:

(i) the points in the discrete neighborhood are considered just as they are generated;
(ii) the points in the discrete neighborhood are ordered with increasing objective function value and considered following this ordering.

In Tables 5 and 6 we report the results obtained by considering as objective function the manufacturing cost and, respectively, the rated efficiency, for every search rule (i), (ii) and every $\bar{d} = 1, 2, \ldots, 5$.

By comparing the results reported in Tables 5 and 6 with the ones of Tables 3 and 4, respectively, it is possible to draw some considerations about the two approaches used in the paper, namely the mixed variable method and the continuous strategy.

First of all, as it can be easily seen, the mixed variable approach, independently from the discrete neighborhood definition ($\bar{d} = i, i = 1, \ldots, 5$) and from the search rule adopted ((i) or (ii)), produces solution points which are uniformly better than the solutions obtained by using the continuous strategy.

Secondly, we note that the best solutions found by algorithm MIVAM for problems $(\tilde{P}^l)$ $(l = 1, 2)$ have an objective function value which is better than the corresponding best solution of problems $(\tilde{R}_h^l)$. At first sight this can be a surprising result especially if we consider that $(\tilde{R}_h^l)$ is obtained from $(\tilde{P}^l)$ by relaxing the

Table 5. Results obtained using the mixed variable approach considering the manufacturing cost

		$\bar{d}=1$		$\bar{d}=2$		$\bar{d}=3$	
	initial	(i)	(ii)	(i)	(ii)	(i)	(ii)
t	1	3	5	5	5	5	5
x_1	17.5	16	16	16	16	16	16
x_2	5.5	6.5	6.5	6.5	6.5	6.5	6.5
x_3	17.8	17.352	16.908	16	17.917	18.088	16
x_4	4	3.596	4.125	4.549	4.138	3.912	4.485
x_5	0.4	0.319	0.307	0.312	0.315	0.306	0.318
x_6	0.5682	0.52	0.533	0.661	0.512	0.529	0.661
y_1	126.6	126.6	126.6	126.6	126.6	126.6	126.6
y_2	160	140	150	140	160	150	140
y_3	200	180	180	200	180	180	200
y_4	1.5727	1.45	1.55	1.5	1.45	1.5	1.5
cost	174.396	148.267	143.677	143.346	144.854	143.473	143.434

		$\bar{d}=4$		$\bar{d}=5$	
	initial	(i)	(ii)	(i)	(ii)
t	1	5	5	5	5
x_1	17.5	16	16	16	
x_2	5.5	6.5	6.5	6.5	6.5
x_3	17.8	17.333	17.348	16.93	16.647
x_4	4	4.25	4.065	4.266	4.526
x_5	0.4	0.31	0.3	0.304	0.315
x_6	0.5682	0.641	0.531	0.64	0.633
y_1	126.6	126.6	126.6	126.6	126.6
y_2	160	140	150	140	140
y_3	200	200	180	200	200
y_4	1.5727	1.4	1.5	1.4	1.45
cost	174.396	142.987	143.05	142.939	143.582

integrality constraints. Nevertheless, this behavior is comprehensible and is essentially due to the fact that Problem ($\tilde{P}^l$) may have many different local minima. As a consequence, a monotone continuous local search, like the derivative free one used in Sect. 3, is more easily attracted toward local minima then algorithm MIVAM which, on the contrary, has some degree of nonmonotonicity due to the growth parameter ξ (see Step 3). This parameter gives to the algorithm a broader capacity of exploring the feasible region thus filtering high local minima and locating better solutions.

The optimization of the manufacturing cost has given rise to a reduction of the active material volume (i.e. stack length and stator wire size), with respect to the initial design, with a “low cost” electric steel type ($t = 5$). On the contrary, the efficiency improvement has been reached not only by increasing the volume but also by choosing a suitable electric steel type ($t = 1$).

Table 6. Results obtained using the mixed variable approach considering the rated efficiency

		$\bar{d}=1$		$\bar{d}=2$		$\bar{d}=3$	
	initial	(i)	(ii)	(i)	(ii)	(i)	(ii)
t	1	1	1	1	1	1	1
x_1	17.5	16	16	16	16	16	16
x_2	5.5	6.5	6.5	6.5	6.5	6.5	6.5
x_3	17.8	18.5	18.5	18.5	18.5	18.5	18.5
x_4	4	3.723	3.982	3.85	3.705	3.5	3.968
x_5	0.4	0.3	0.3	0.3	0.3	0.3	0.3
x_6	0.568	0.583	0.582	0.603	0.572	0.545	0.578
y_1	126.6	126.6	126.6	126.6	126.6	126.6	126.6
y_2	160	160	190	160	180	190	190
y_3	200	220	220	220	220	220	220
y_4	1.5727	1.75	1.75	1.75	1.75	1.75	1.75
efficiency	88.30%	91.09%	91.29%	91.08%	91.26%	91.33%	91.29%
		$\bar{d}=4$		$\bar{d}=5$			
	initial	(i)	(ii)	(i)	(ii)		
t	1	1	1	1	1		
x_1	17.5	16	16	16	16		
x_2	5.5	6.5	6.5	6.5	6.5		
x_3	17.8	18.5	18.5	18.5	18.5		
x_4	4	3.5	3.859	4.084	3.705		
x_5	0.4	0.3	0.3	0.3	0.3		
x_6	0.568	0.545	0.567	0.589	0.572		
y_1	126.6	126.6	126.6	126.6	126.6		
y_2	160	190	190	190	180		
y_3	200	220	220	220	220		
y_4	1.5727	1.75	1.75	1.75	1.75		
efficiency	88.30%	91.33%	91.30%	91.23%	91.26%		

There is another issue of the mixed variable strategy which deserves some attention. In fact, in spite of the expectations, it is possible to note that there is no direct connection between the dimension of the discrete neighborhood (i.e. $\bar{d} = i$, $i = 1, \ldots, 5$) and the objective function value of the corresponding solution. In particular, the manufacturing cost (rated efficiency) of the solution point does not decrease (increase) as the discrete neighborhood dimension grows. Moreover, as concerns the manufacturing cost, we remark that, for search rule (ii), the best solution is obtained by taking $\bar{d} = 4$. As for the efficiency, if we consider search rule (i), the best solution is obtained when $\bar{d} = 2$.

To conclude, the best solution with respect to the manufacturing cost is obtained by setting $\bar{d} = 5$ and search rule (i), while for the rated efficiency the best solution is obtained by setting $\bar{d} = 3$ and search rule (i).

Finally, it is important to note that all the runs, both for the continuous approach and for the mixed variable strategy, required a low computational cost in terms of CPU time which never exceeded a few seconds.

6 Concluding remarks

In this paper the optimal design of induction motors has been considered. In particular, two different solution strategies have been analyzed and compared. The obtained results are very interesting and show the fruitfulness of directly taking into account the presence of both continuous and discrete variables.

It is important to underline that the optimized designs require low additional tooling costs and this aspect is very important from the manufacturer point of view. Moreover, the results show the effectiveness of the proposed approach and should stimulate the designers towards the frequent use of this powerful tool for the design optimization of electric motors.

However, the theoretical results regarding algorithms MIVAM and DFA reported in [3] and [4] respectively, do not apply in this context. In fact, problems $(\tilde{P}^l)$ and $(\tilde{R}^l_h)$ for $l = 1, 2$ and $h = 0, 1, \ldots, 5$ do not satisfy the assumptions required to prove the convergence of the considered methods in that the objective functions $\tilde{f}^{(l)}(x, y, t)$ for $l = 1, 2$ are not continuous on the feasible region. This problem is due to the nonlinear constraints handling by means of the elimination technique introduced in Sect. 2. Thus, to overcome this difficulty it would be desirable to define a better strategy to handle the nonlinear constraints. Moreover, the numerical results point out the presence of many different local minima of problems $(\tilde{P}^l)$ for $l = 1, 2$ and this would suggest the opportunity of defining a mixed variable strategy which can take into account the global aspect of the problem.

All the previous issues are subject of continuing research.

References

[1] Audet C, Dennis JE (2001) Pattern Search Algorithms for Mixed Variable Programming. *SIAM Journal on Optimization* **11**: 573–594.
[2] Chiricozzi E, Parasiliti F, Villani M (1997) Experience in design optimization technique of high efficiency induction motors. In: *Energy Efficiency Improvements in Electric Motors and Drives*, pp 116–139. Springer, Berlin Heidelberg New York.
[3] Lucidi S, Piccialli V, Sciandrone M (2002) Algorithm Model for Mixed Variable Programming. *Tech. Rep. DIS 17-02*, University of Rome “La Sapienza”.
[4] Lucidi S, Sciandrone M (2002) A derivative-free algorithm for bound constrained optimization. *Computational Optimization and Applications* **21**(2): 119–142.
[5] Stoneking D, Bilbro G, Trew R, Gilmore P, Kelley CT (1992) Yield Optimization Using a GaAs Process Simulator Coupled to a Physical Device Model. *IEEE Transactions on Microwave Theory and Techniques* **40**: 1353–1363.
[6] Pardalos PM, Resende M (2002) Handbook of Applied Optimization. Oxford University Press